MATH 2060 Tutorial 11

Section 9.2

Q4 Discuss the convergence or divergence of the series with n-th term

(a) $a_n = 2^n e^{-n}$ (f) $b_n = n!\, e^{-n^2}$

Sol:

(a) Recall the Ratio Test:

(i) If $\exists\ r<1$ and $K \in \mathbb{N}$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \le r$, $\forall n \ge K$.

then Σx_n is absolutely convergent

(ii) If $\exists\ K \in \mathbb{N}$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \ge 1$, $\forall n \ge K$,

then Σx_n is divergent

Here we can apply the Ratio Test:

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{e} < 1$ for all n

Therefore, Σa_n is convergent

(f) Here we can still apply the Ratio Test:

$\left|\frac{b_{n+1}}{b_n}\right| = (n+1)e^{-(2n+1)}$

Let $f(x) = (x+1)e^{-(2x+1)}$ and note that

$f'(x) = e^{-(2x+1)} - 2(x+1)e^{-(2x+1)} = -(2x+1)e^{-(2x+1)} < 0$, $\forall x \ge 1$

Hence, $f(n) \le f(1) = \frac{2}{e^3}$ for all $n \ge 1$,

which implies $\left|\frac{b_{n+1}}{b_n}\right| \le \frac{2}{e^3} < 1$, $\forall n$

Then Σb_n is convergent

8. Let $0 < a < 1$ and consider the series

$$a^2 + a + a^4 + a^3 + \cdots + a^{2n} + a^{2n-1} + \cdots.$$

Show that the Root Test applies, but that the Ratio Test does not apply.

Sol: Recall the Root Test :

(i) If $\exists\, r<1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{\frac{1}{n}} \le r$, $\forall\, n \ge K$,

then Σx_n is absolutely convergent

(ii) If $\exists\, K \in \mathbb{N}$ s.t. $|x_n|^{\frac{1}{n}} \geq 1$, $\forall\, n \geq K$,

then the series Σx_n is divergent

Here we first apply the Root Test to prove the convergence of the series

Note that we can write the series with the n-th term as

$$x_n = \begin{cases} a^{n+1} & \text{if } n \text{ is odd} \\ a^{n-1} & \text{if } n \text{ is even} \end{cases}$$

which is a rearrangement of the series $\sum_{n=1}^{\infty} a^n$

Then $|x_n|^{\frac{1}{n}} = \begin{cases} a^{1+\frac{1}{n}} & \text{if } n \text{ is odd} \\ a^{1-\frac{1}{n}} & \text{if } n \text{ is even} \end{cases}$

And it's cleer that $\lim_{n\to\infty} |x_n|^{\frac{1}{n}} = a < 1$

By the Root Test, Σx_n is convergent

However, if we want to apply the Ratio Test :

$$\left|\frac{x_{n+1}}{x_n}\right| = \begin{cases} a^{-1} > 1 & \text{if } n \text{ is odd} \\ a^{3} < 1 & \text{if } n \text{ is even} \end{cases}$$

Ratio Test does not apply in the sense that one cannot find $K \in \mathbb{N}$ and $r \in [0,1)$ such that for all $n \geq K$,

either $\left|\frac{x_{n+1}}{x_n}\right| \leq r$ or $\left|\frac{x_{n+1}}{x_n}\right| \geq 1$

17. If $p > 0$, $q > 0$, show that the series

$$\sum \frac{(p+1)(p+2)\cdots(p+n)}{(q+1)(q+2)\cdots(q+n)}$$

converges for $q > p + 1$ and diverges for $q \leq p + 1$.

Sol: Recall Raabe's Test:

(i) If there exists $\alpha > 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{\alpha}{n} \quad \text{for all } n \geq K,$$

then $\sum x_n$ is absolutely convergent

(ii) If there exist $\alpha \leq 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{\alpha}{n} \quad \text{for all } n \geq K,$$

then $\sum x_n$ is not absolutely convergent

Here, we let $x_n = \frac{(p+1)(p+2)\cdots(p+n)}{(q+1)(q+2)\cdots(q+n)} > 0$ and wish to apply the Raabe's Test

Note that $\left|\frac{x_{n+1}}{x_n}\right| = \frac{p+n+1}{q+n+1}$

Then $\lim_{n\to\infty} \left[n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)\right] = \lim_{n\to\infty} \frac{n(q-p)}{q+n+1} = q-p$

① If $q > p+1$, then $q-p > 1$ and $\left|\frac{x_{n+1}}{x_n}\right| = \frac{p+n+1}{q+n+1}$ is increasing as n increases

Then $n\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right) \geq q-p$, i.e. $\left|\frac{x_{n+1}}{x_n}\right| \leq 1-\frac{q-p}{n}$

Take $\alpha = q-p$ in Raabe's Test, we conclude $\sum x_n$ converges

② If $q < p+1$, then $q-p<1$, by Cor 9.2.9

$\sum x_n$ is divergent ($x_n > 0$ for all n)

③ If $q = p+1$, then $x_n = \frac{p+1}{p+1+n}$

Since $p>0$, $\frac{p+1}{p+1+n} > \frac{1}{n+1}$

Now let $y_n = \frac{1}{n+1}$, which is a divergent series

And by Comparison Test, it follows that $\sum x_n$ is also divergent